

ATTRACTOR BIFURCATION AND FINAL PATTERNS OF THE N-DIMENSIONAL AND GENERALIZED SWIFT-HOHENBERG EQUATIONS

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ABSTRACT. In this paper I will investigate the bifurcation and asymptotic behavior of solutions of the Swift-Hohenberg equation and the generalized Swift-Hohenberg equation with the Dirichlet boundary condition on a one-dimensional domain $(0, L)$. I will also study the bifurcation and stability of patterns in the n -dimensional Swift-Hohenberg equation with the odd-periodic and periodic boundary conditions. It is shown that each equation bifurcates from the trivial solution to an attractor \mathcal{A}_λ when the control parameter λ crosses λ_c , the principal eigenvalue of $(I + \Delta)^2$. The local behavior of solutions and their bifurcation to an invariant set near higher eigenvalues are analyzed as well.

1. Introduction. Pattern formation is an interesting phenomenon which is often observed in physics and chemistry. A physical system when driven sufficiently far from equilibrium tends to form geometric patterns [5]. This phenomena is determined by nonlinear aspects of the system under study.

To study how those patterns form and evolve is the subject of “non-equilibrium physics”. Non-equilibrium phenomena include Taylor-Couette flow, parametric waves, reaction-diffusion systems, propagation of electromagnetic waves in certain types of media, and convection. Convection is a widely-studied example of the dynamics that can occur in a system under the influence of a constant, homogeneous temperature gradient. The mechanism of convection is responsible for many phenomena of great interest, such as cloud formation, ocean currents, plate tectonics, and crystal growth.

In the study of spatial patterns an important role is played by model equations. A model equation, while simpler than full system of equations, captures most features which control the pattern formation phenomenon of the system. Recently attention has been drawn to the study of fourth-order model equations involving bistable dynamics. The fourth order pattern forming equation is of central importance [6, 7]:

$$\frac{du}{dt} + \frac{\partial^4 u}{\partial x^4} + q \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0,$$

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where q measures the pattern forming tendency. An extensive study of the stationary equation, known as the Symmetric Bistable (SBS) equation, can be found in [15]. As discussed in the latter, stationary equations of a variety of fourth order model equations such as the extended Fisher-Kolmogorov (EFK) equation, the Swift-Hohenberg (SH) equation, the Suspension Bridge equation, Bretherton's equation, the nonlinear Schrodinger equation can be scaled to the Symmetric Bistable (SBS) equation (or canonical equation). This can be put in a different way too. In fact the SBS equation

$$\frac{du}{dt} + A \frac{\partial^4 u}{\partial x^4} + B \frac{\partial^2 u}{\partial x^2} + Cu + u^3 = 0, \quad A, B, C > 0;$$

can be written as

$$\frac{1}{C} \frac{du}{dt} + \left(\frac{\partial^2}{\partial x^2} + \frac{B}{2A} I \right)^2 u + \left(\frac{C}{A} - \left(\frac{B}{2A} \right)^2 \right) u + \frac{1}{C} u^3 = 0.$$

and the latter can be scaled to the dimensionless form of the Swift-Hohenberg equation which shall be the main focus of this paper.

The Swift-Hohenberg (SH) equation

$$u_t = - \left(\frac{\partial^2}{\partial x^2} + I \right)^2 u + \lambda u - u^3, \quad x \in (0, L), \quad \lambda \in \mathbb{R}.$$

was proposed in 1977 in connection with Rayleigh-Bénard's convection. Although the SH equation cannot be derived systematically from the Boussinesq equations, it captures much of the observed physical behavior and has now become a general tool used to investigate not only Rayleigh-Bénard convection, but also other pattern-forming systems [5].

In this paper we will study qualitatively the bifurcation problem of the 1-dimensional Swift-Hohenberg (SH) equation and generalized Swift-Hohenberg (GSH) equation with the Dirichlet boundary condition. Some numerical results, compatible with our results concerning the SH equation with the Dirichlet boundary condition, can be found in [14].

The main technical tool is the new bifurcation theory developed recently by Ma and Wang [12]. The theory is based on a new notion of bifurcation called attractor bifurcation. The main theorem associated with the attractor bifurcation states that under certain conditions and provided the critical state is asymptotically stable, when the first eigenvalue of the linearized equation crosses the imaginary axis, the system bifurcates from a trivial steady state solution to an attractor with dimension between $m - 1$ and m , where m is the algebraic multiplicity of the first eigenvalue. Another important ingredient of the analysis is the reduction of the equation to its central manifold. This involves lengthy calculations and careful examination of the nonlinear interaction of higher order. The key idea is to derive an higher order non-degenerate approximation which is sufficient for the bifurcation analysis.

The main results obtained in this article can be summarized as follows. First, we have shown that as the parameter λ crosses the first critical value λ_c , the Swift-Hohenberg equation bifurcates from the trivial solution to an attractor \mathcal{A}_λ , with a dimension between $m - 1$ and m depending on boundary conditions, where m is the multiplicity of the first eigenvalue of the linearized problem. We will determine the critical number λ_c precisely. Second, as an attractor, the bifurcated attractor \mathcal{A}_λ has asymptotic stability in the sense that it attracts all solutions with initial data in the phase space outside of the stable manifold, with codimension m , of the trivial solution. Third, the 1-d SH equation with the Dirichlet boundary condition or

odd periodic boundary condition bifurcates to exactly two steady state solutions; and with the periodic boundary condition, to an attractor homeomorphic to S^1 . Finally, in n -dimensional ($n \leq 3$) case with the odd periodic boundary condition, the number of steady state solutions contained in the bifurcated attractor \mathcal{A}_λ , for $\lambda > \lambda_c$, has been given precisely; and with the periodic boundary condition, it is shown that the bifurcated attractor \mathcal{A}_λ contains a torus \mathbb{T}^n which consists of steady state solutions.

It is worth mentioning that although the problem has certain symmetry, the method used in the article is crucial for proving the bifurcated object is precisely the attractor and for proving the stability property of the bifurcated object.

The paper is organized as follows. In section 2, we recall the main results of the attractor bifurcation and center manifold reduction method. In Section 3, we state and prove our main theorems concerning the bifurcation of the Swift-Hohenberg equation and the generalized Swift-Hohenberg equation with the Dirichlet boundary condition. In section four, the main results concerning the n -dimensional Swift-Hohenberg equation with the odd-periodic and periodic boundary condition shall be stated.

2. Abstract Bifurcation Theory and Reduction Method.

2.1. Attractor Bifurcation Theorem. Here we shall recall some results of dynamic bifurcation of abstract nonlinear evolution equations developed in [12]. Also we will refer the reader to [13] for a comprehensive study of the dynamic bifurcation theory developed by Ma and Wang.

Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. Consider the following nonlinear evolution equation

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \quad (1)$$

$$u(0) = u_0, \quad (2)$$

where $u : [0, \infty) \rightarrow H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \rightarrow H$ are parameterized linear completely continuous fields continuously depending on $\lambda \in \mathbb{R}$, which satisfy

$$\begin{cases} L_\lambda = -A + B_\lambda & \text{a sectorial operator,} \\ A : H_1 \rightarrow H & \text{a linear homeomorphism,} \\ B_\lambda : H_1 \rightarrow H & \text{the parameterized linear compact operators.} \end{cases} \quad (3)$$

We can see that L_λ generates an analytic semi-group $\{e^{-tL_\lambda}\}_{t \geq 0}$ and then we can define fractional power operators L_λ^α for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_2 < \alpha_1$, and $H_0 = H$.

We now assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha \rightarrow H$ for some $0 \leq \alpha < 1$ are a family of parameterized C^r bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}$, such that

$$G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall \lambda \in \mathbb{R}. \quad (4)$$

For the linear operator A we assume that there exists a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}$ and an eigenvector sequence $\{e_k\} \subset H_1$, i.e.,

$$\begin{cases} Ae_k = \rho_k e_k, \\ 0 < \rho_1 \leq \rho_2 \leq \cdots, \\ \rho_k \rightarrow \infty \ (k \rightarrow \infty) \end{cases} \quad (5)$$

where $\{e_k\}$ is an orthogonal basis of H .

For the compact operator $B_\lambda : H_1 \rightarrow H$, we also assume that there is a constant $0 < \theta < 1$ such that

$$B_\lambda : H_\theta \longrightarrow H \text{ bounded, } \forall \lambda \in \mathbb{R}. \quad (6)$$

We know that the operator $L = -A + B_\lambda$ satisfying (5) and (6) is a sectorial operator. It generates an analytic semigroup $\{S_\lambda(t)\}_{t \geq 0}$. Then the solution of (1) and (2) can be expressed as

$$u(t, u_0) = S_\lambda(t)u_0, \quad t \geq 0.$$

Definition 1. A set $\Sigma \subset H$ is called an invariant set of (1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (1) is said to be an attractor if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$ we have

$$\lim_{t \rightarrow \infty} \text{dist}_H(u(t, \varphi), \Sigma) = 0. \quad (7)$$

The largest open set U satisfying (7) is called the basin of attraction of Σ .

Definition 2. 1. We say that the equation (1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ an invariant set Ω_λ , if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (1), $0 \notin \Omega_{\lambda_n}$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n &= \lambda_0, \\ \lim_{n \rightarrow \infty} \max_{x \in \Omega_{\lambda_n}} |x| &= 0. \end{aligned}$$

2. If the invariant sets Ω_λ are attractors of (1), then the bifurcation is called attractor bifurcation.
3. If Ω_λ are attractors and are homotopy equivalent to an m -dimensional sphere S^m , then the bifurcation is called S^m -attractor bifurcation.

The following dynamic bifurcation theorem for (1) was proved in [12].

Theorem 1 (Attractor Bifurcation Theorem). *Assume that (3)–(6) hold. Let the eigenvalues (counting multiplicity) of L_λ be given by $\beta_1(\lambda), \beta_2(\lambda), \dots, \beta_k(\lambda), \dots \in \mathbb{C}$. Suppose that*

$$\text{Re} \beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases} \quad (1 \leq i \leq m), \quad (8)$$

$$\text{Re} \beta_j(\lambda_0) < 0 \quad (m+1 \leq j). \quad (9)$$

Let the eigenspace of L_λ at λ_0 be

$$E_0 = \bigcup_{i=1}^m \{u \in H_1 \mid (L_{\lambda_0} - \beta_i(\lambda_0))^k u = 0, k = 1, 2, \dots\}.$$

and $u = 0$ be a locally asymptotically stable equilibrium point of (1) at $\lambda = \lambda_0$. Then the following assertions hold.

1. The equation (1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to an attractor \mathcal{A}_λ for $\lambda > \lambda_0$, with $m - 1 \leq \dim \mathcal{A}_\lambda \leq m$, which is connected if $m > 1$.
2. The attractor \mathcal{A}_λ is a limit of a sequence of m -dimensional annulus M_k with $M_{k+1} \subset M_k$; in particular if \mathcal{A}_λ is a finite simplicial complex, then \mathcal{A}_λ has the homotopy type of S^{m-1} .
3. For any $u_\lambda \in \mathcal{A}_\lambda$, u_λ can be expressed as

$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \quad v_\lambda \in E_0.$$

4. If the number of equilibrium points of (1) in \mathcal{A}_λ is finite, then we have the following index formula

$$\sum_{u_i \in \mathcal{A}_\lambda} \text{ind}[-(L_\lambda + G), u_i] = \begin{cases} 2 & \text{if } m = \text{odd}, \\ 0 & \text{if } m = \text{even}. \end{cases}$$

5. If $u = 0$ is globally stable for (1) at $\lambda = \lambda_0$, then for any bounded open set $U \subset H$ with $0 \in U$, there is an $\varepsilon > 0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor \mathcal{A}_λ bifurcated from $(0, \lambda_0)$ attracts $U \setminus \Gamma$ in H , where Γ is the stable manifold of $u = 0$ with codimension m . In particular, if (1) has a global attractor for all λ near λ_0 , then ε can be chosen independently of U .

The following theorem is an immediate result of the attractor bifurcation theorem:

Theorem 2 (Pitchfork bifurcation). *If the first eigenvalue is simple, i.e. $m = 1$, then the bifurcated attractor \mathcal{A}_λ consists of exactly two points u_1 and u_2 . Moreover, for any bounded open set $U \subset H$ with $0 \in U$ there is an $\varepsilon > 0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, U can be decomposed into two open sets U_1^λ and U_2^λ satisfying such that*

1. $\bar{U} = \bar{U}_1^\lambda + \bar{U}_2^\lambda, \bar{U}_1^\lambda \cap \bar{U}_2^\lambda = \emptyset$ and $0 \in \partial U_1^\lambda \cap \partial U_2^\lambda$,
2. $u_i^\lambda \in U_i^\lambda (i = 1, 2)$, and
3. $\lim_{t \rightarrow \infty} \|u(t, \varphi) - u_i^\lambda\|_H = 0$,
for any $\varphi \in U_i^\lambda (i = 1, 2)$, where $u(t, \varphi)$ is the solution of (1).

The following theorem will be useful later.

Theorem 3 ([13]). *Let v be a two-dimensional C^r ($r \geq 1$) vector field given by*

$$v_\lambda = \lambda x - G_k(x, \lambda) + o(|x|^k),$$

where $x \in \mathbb{R}^2$, G_k is a k -multilinear field, and $k = 2m + 1$ ($m \geq 1$). If G_k satisfies

$$C_1|x|^{k+1} \leq G_k(x, \lambda), x \geq C_2|x|^{k+1},$$

for some constant $C_2 > C_1 > 0$, then v_λ bifurcates from $(x, \lambda) = (0, 0)$ on $\lambda > 0$ to an attractor Ω_λ , which is homeomorphic to S^1 . Moreover, one and only one of the following is true:

1. Ω_λ is a periodic orbit.
2. Ω_λ consists of only singular points.
3. Ω_λ contains at most $2(k+1) = 4(m+1)$ singular points, and has $4N + n(N + n \geq 1)$ singular points. $2N$ of which are saddle points, $2N$ of which are stable node points (possibly degenerate), and n of which have index zero.

2.2. Reduction Method. A useful tool in the study of bifurcation problems is the reduction of the equation to its local center manifold. The idea is to project the equation to a finite dimensional space after a change of basis. An extensive study of the method of reduction to center manifold can be found in [13].

Consider the following non-linear evolution equation:

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda), \\ u(0) = u_0. \end{cases} \quad (10)$$

with $L_\lambda = -A + B_\lambda : H_1 \longrightarrow H$ being a symmetric linear continuous field, $G(\cdot, \lambda) : H_1 \longrightarrow H$ being a C^∞ , which can be expressed as

$$G(u, \lambda) = \sum_{n=k}^{\infty} G_n(u, \lambda), \quad \text{for some } k \geq 2$$

where $G_n : H_1 \times \dots \times H_1 \longrightarrow H$ is an n -multiple linear mapping, and $G_n(u, \lambda) = G_n(u, \dots, u, \lambda)$.

Let $\beta_i(\lambda)$ and $e_i(\lambda)$ be the eigenvalue and eigenvector of L_λ respectively. Since L_λ is symmetric, $\beta_i(\lambda)$'s are real. Assume that $e_i(\lambda)$'s form an orthogonal basis for the space H . Also, assume the following conditions hold true:

$$\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases} \quad (1 \leq i \leq m), \quad (11)$$

$$\begin{cases} \beta_j(\lambda_0) > 0 & \text{for } m < j \leq m+n, \\ \beta_j(\lambda_0) < 0 & \text{for } m+n < j. \end{cases} \quad (12)$$

then with a change of basis, the equation can be written in the new basis (one can think of this as Fourier series). After projection to the subspace generated by the first eigenvalue, the equation can be reduced to the central manifold as follows:

$$\frac{dx}{dt} = J_{m\lambda}x + g(x, \lambda), \quad (13)$$

where $x = (x_1, \dots, x_m)^t$, $J_{m\lambda}$ is the Jordan matrix corresponding to the first m eigenvalues of L_λ , and $g(x, \lambda) = (g_1(x, \lambda), \dots, g_m(x, \lambda))^t$ with $g_i(x, \lambda) = \langle G(x + \phi(x, \lambda), e_i), e_i \rangle$. Here $\phi(x, \lambda)$ is the center manifold function near λ_0 . Finally the above equation can be rewritten as follows:

$$\frac{dx}{dt} = J_{m\lambda}x + \sum_{p=k}^{k+N-1} F_p(x) + o(x^{N+k-1}), \quad N \geq 1. \quad (14)$$

This last equation is called the N^{th} -order approximation of the (13)

3. Bifurcation of The SH and The GSH Equations.

3.1. The SH and The GSH Equations. The following nonlinear equation

$$\begin{cases} u_t = -(\frac{\partial^2}{\partial x^2} + I)^2 u + \lambda u - u^3 + \mu u^2, & \lambda \in \mathbb{R}, x \in (0, L); \\ u(0, t) = u(L, t) = u''(0, t) = u''(L, t) = 0; \\ u(x, 0) = u_0(x). \end{cases} \quad (15)$$

is known as the Swift-Hohenberg (SH) equation when $\mu = 0$, and as the generalized Swift-Hohenberg (GSH) when $\mu > 0$. The SH equation, proposed in 1977 by Swift and Hohenberg [17], has been shown to be a useful tool in the study of a variety of problems, such as Taylor-Couette flow [10, 16], and in the study of lasers [11]. The GSH equation was proposed later in connection with the study of localized patterns. Extensive numerical and analytical studies have been done on both equations; for example see [4, 5, 2, 15] for the SH equation ; and for the GSH equation see [1, 3, 9, 8].

The existence of solutions of (15) is established in [15] using a topological shooting method. Some bifurcation analyses and some numerical simulations, when $0 < \lambda < 1$, are conducted in [14].

3.2. Functional settings. Now we employ tools introduced in the second section to discuss the bifurcation of the SH and GSH equation with the Dirichlet condition. In an appropriate functional setting the equation 15 can be expressed in the following form

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda), \\ u(0) = u_0, \end{cases} \quad (16)$$

where the operators L_λ and G are defined as follows:

$$\begin{aligned} L_\lambda &= -A + B_\lambda : H_1 \hookrightarrow H, \\ A &= (I + \frac{\partial^2 u}{\partial x^2})^2 : H_1 \hookrightarrow H, \\ B_\lambda &= \lambda I : H_1 \hookrightarrow H, \\ Gu &= \mu u^2 - u^3 : H_1 \hookrightarrow H. \end{aligned} \quad (17)$$

with Hilbert spaces H and H_1 :

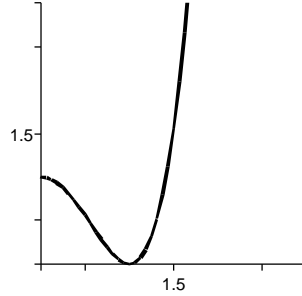
$$\begin{aligned} H_1 &= \{u \in H^4(0, L) \mid u, u'' = 0 \text{ at } x = 0, L\}, \\ H &= L^2(0, L). \end{aligned} \quad (18)$$

Since H_1 is compactly imbedded in H , $H_1 \hookrightarrow H$, it is clear that $B : H_1 \hookrightarrow H$ is a compact operator. With an easy calculation one can see the eigenvalues of $A : H_1 \hookrightarrow H$ are $\lambda_n = (1 - (\frac{n\pi}{L})^2)^2$; hence, assuming $L \neq n\pi$ for any $n \in \mathbb{N}$, A is a homeomorphism. Therefore, $L_\lambda = -A + B$ is a completely continuous field. When $L = n\pi$ for some integer n , $A + I$ will be a homeomorphism. In this case, L_λ is still a completely continuous field, for L_λ can be written as the sum of $A + I$ and $(\lambda - 1)I$.

3.3. Bifurcation of The SH and The GSH equations. In general the eigenvalues of $A = (I + \frac{\partial^2}{\partial x^2})^2 : H_1 \hookrightarrow H$ are $\lambda_n = P(\frac{n\pi}{L})$, where $P(x) = (1 - x^2)^2$ (see Figure1), and its principal eigenvalue is $\lambda_c = \min\{ P(\frac{n\pi}{L}) \mid n \geq 1 \}$. Therefore, depending on the value of L , we might get a different critical value and a different final profile associated with it.

Theorem 4. *Let λ_c be the principal eigenvalue of $(I + \frac{\partial^2 u}{\partial x^2})^2 : H_1 \hookrightarrow H$. Then the following assertions hold true for the SH equation with the Dirichlet boundary condition (eq. 15 with $\mu = 0$):*

1. *For $\lambda \leq \lambda_c$, the trivial solution $u = 0$ is globally asymptotically stable (Figure 2).*

FIGURE 1. The graph of $P(x) = (1 - x^2)^2$.

2. For $\lambda > \lambda_c$ The SH-d bifurcates from $(0, \lambda_c)$ to an attractor bifurcation \mathcal{A}_λ which consists of exactly two steady states (Figure 2).
3. For $\lambda > \lambda_c$, the bifurcated attractor \mathcal{A}_λ consists of exactly two steady states u_1^λ, u_2^λ given by

$$\begin{cases} u_1^\lambda = \beta(\lambda)\phi_c + o(|\beta(\lambda)|), \\ u_2^\lambda = -\beta(\lambda)\phi_c + o(|\beta(\lambda)|), \\ \beta(\lambda) = \sqrt{\frac{\beta_c(\lambda)}{\alpha}}; \end{cases} \quad (19)$$

where $\beta_c(\lambda)$ is the first eigenvalue of the linearized equation, ϕ_c the normalized eigenvector associated with it, and $\alpha = \langle (\phi_c)^3, \phi_c \rangle_H$.

4. The stable manifold $\Gamma \subset H$ of $u = 0$ separates the phase space H into two open sets U_1^λ and U_2^λ , where $\lambda_c < \lambda < \lambda_c + \epsilon$ for some $\epsilon > 0$, which are the basin of attraction of u_1^λ and u_2^λ respectively, i.e.

$$\begin{aligned} H &= \overline{U_1^\lambda} + \overline{U_2^\lambda}, \\ U_1^\lambda \cap U_2^\lambda &= \emptyset, \\ \partial U_1^\lambda \cap \partial U_2^\lambda &= \Gamma, \\ u_i^\lambda &\in U_i^\lambda \quad i = 1, 2, \\ \lim_{t \rightarrow \infty} \|u(t, \varphi) - u_i^\lambda\|_H &= 0, \end{aligned}$$

for any $\varphi \in U_i^\lambda (i = 1, 2)$, where $u(t, \varphi)$ is the solution of the SH equation.

5. For any integer n , the SH equation bifurcates from $(u, \lambda) = (0, \lambda_n)$, on $\lambda > \lambda_n$, to an attractor consisting of two steady state solutions of the SH equation.

For instance, when $L \leq \pi$, the final patterns are given by

$$\begin{cases} u_1^\lambda = \beta(\lambda) \sin \frac{\pi x}{L} + o(|\beta(\lambda)|), \\ u_2^\lambda = -\beta(\lambda) \sin \frac{\pi x}{L} + o(|\beta(\lambda)|), \\ \beta(\lambda) = \sqrt{\frac{4}{3}(\lambda - (1 - (\frac{\pi}{L})^2)^2)}. \end{cases} \quad (20)$$

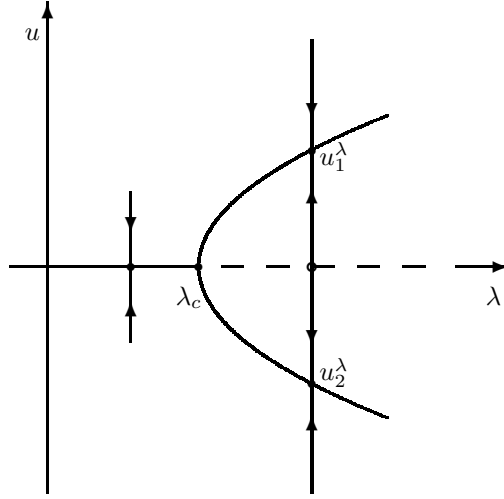


Figure2. Pitchfork bifurcation of the SH equation with the Dirichlet boundary condition.

Proof. For the proof of the first part see Theorem (6). In fact the existence of the global attractor for the SH equation is known [14].

Without loss of generality, we will assume that $L < \pi$. The eigenvectors and eigenvalues of $L_\lambda : H_1 \hookrightarrow H$ are known to be:

$$\begin{aligned}\beta_n(\lambda) &= \lambda - (1 - (\frac{n\pi}{L})^2)^2, \\ \phi_n(x) &= \sqrt{\frac{2}{L}} \sin(\frac{n\pi}{L}x).\end{aligned}\tag{21}$$

Moreover, the eigenvalues have the following properties:

$$\beta_1(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_1, \\ = 0 & \text{if } \lambda = \lambda_1, \\ > 0 & \text{if } \lambda > \lambda_1, \end{cases}$$

$$\beta_n(\lambda_1) < 0 \quad \forall n \neq 1.$$

Hence this theorem is a direct result of the pitchfork bifurcation theorem(2). We only need to prove (20). This can be proven by the Lyapunov-Schmidt reduction method near $\lambda_c = \lambda_1$. Let $u \in H$ and $u = \sum_{k=1}^{\infty} x_k \phi_k(x)$. Then the steady state bifurcation equation of the SH equation can be expressed as

$$\beta_n x_n - \sqrt{\frac{2}{L}} \int_0^L u^3 \sin(\frac{n\pi x}{L}) dx = 0,$$

where $\beta_n = \lambda - (1 - (\frac{n\pi}{L})^2)^2$. We have

$$\begin{aligned}u^3 &= \sqrt{(\frac{2}{L})^3} \frac{1}{4} \sum_{j,k,l \in \mathbb{N}} x_j x_k x_l [\sin(j+k-l) \frac{\pi x}{L} + \sin(j+l-k) \frac{\pi x}{L} + \\ &\quad \sin(k+l-j) \frac{\pi x}{L} - \sin(j+k+l) \frac{\pi x}{L}];\end{aligned}$$

so we will have

$$\begin{aligned}
x_1 &= \frac{1}{2L\beta_1} [3 \sum_{\substack{j \in \mathbb{N} \\ k+l=1+j}} x_j x_k x_l], \\
x_2 &= \frac{1}{2L\beta_2} [3 \sum_{\substack{j \in \mathbb{N} \\ k+l=2+j}} x_j x_k x_l], \\
x_3 &= \frac{1}{2L\beta_3} [3 \sum_{\substack{j \in \mathbb{N} \\ k+l=3+j}} x_j x_k x_l - x_1^3], \\
x_n &= \frac{1}{2L\beta_n} [3 \sum_{\substack{j \in \mathbb{N} \\ k+l=n+j}} x_j x_k x_l - \sum_{\substack{j,k,l \in \mathbb{N} \\ j+k+l=n}} x_j x_k x_l], \quad n \geq 4.
\end{aligned}$$

Hence, by induction, we have:

$$\begin{aligned}
x_2 &= o(|x_1|^3), \\
x_3 &= -\frac{1}{2L\beta_3} x_1^3 + o(|x_1|^3), \\
x_n &= c_n x_1^n + o(|x_1|^n),
\end{aligned}$$

for $n \geq 4$, where c_n is a constant. Therefore, we have the following bifurcation equation for the SH equation:

$$2L\beta_1 x_1 - 3x_1^3 + o(|x_1|^3) = 0. \quad (22)$$

This completes the proof. The proof of the last part of the theorem is similar. \square

For the generalized Swift-Hohenberg (GSH) equation, the stability of the trivial solution $u = 0$ may not be true at λ_c ; nevertheless, we can prove the existence of a bifurcation near the critical value.

Theorem 5. *The following assertion hold true for the GSH equation (eq. 15 when $\mu > 0$) with Dirichlet boundary conditions :*

1. (15) bifurcates from $(0, \lambda_c)$ to a unique saddle point u^λ (with Morse index one) on $\lambda < \lambda_c$, and to a unique attractor u^λ on $\lambda > \lambda_c$.
2. If $\lambda > \lambda_c$ there is an open set U of $u = 0$ which is divided into two open sets by the stable manifold Γ of $u = 0$ with codimension one in H .

$$\begin{aligned}
\overline{U} &= \overline{U_1^\lambda} + \overline{U_2^\lambda}, \\
U_1^\lambda \cap U_2^\lambda &= \emptyset, \\
\partial U_1^\lambda \cap \partial U_2^\lambda &= \Gamma, \\
u^\lambda &\in U_1^\lambda, \text{ and} \\
\lim_{t \rightarrow \infty} \|u(t, \varphi) - u^\lambda\|_H &= 0,
\end{aligned}$$

for any $\varphi \in U_1^\lambda$, where $u(t, \varphi)$ is the solution of (15).

3. Near λ_c , the bifurcated singular points u_λ can be expressed as

$$u^\lambda = \beta(\lambda) \phi_c + o(|\beta(\lambda)|).$$

where $\beta_c(\lambda)$ is the first eigenvalue of the linearized equation, ϕ_c the normalized eigenvector corresponding to it, $\alpha = \langle (\phi_c)^2, \phi_c \rangle_H$, $\beta(\lambda) = -\frac{\beta_c}{\mu\alpha}$.

For instance when $L < \pi$, that is $\lambda_c = (1 - (\frac{\pi}{L})^2)^2$, we have $\alpha = \frac{8\sqrt{2}}{3\pi\sqrt{L}}$ and

$$u^\lambda = \frac{3\pi}{8\mu}(\lambda_1 - \lambda) \sin(\frac{\pi}{L}x) + o(|\lambda_1 - \lambda|).$$

Proof. By reducing the equation to its center manifold near λ_c , we get the following equation when $\mu > 0$:

$$\frac{dx_c}{dt} = \beta_c x_1 + \alpha \mu x_c^2 + o(x_c^2).$$

where $\alpha = < (\phi_c)^3, \phi_c >_H$ is a constant. Then the theorem is an obvious result of this reduced equation. \square

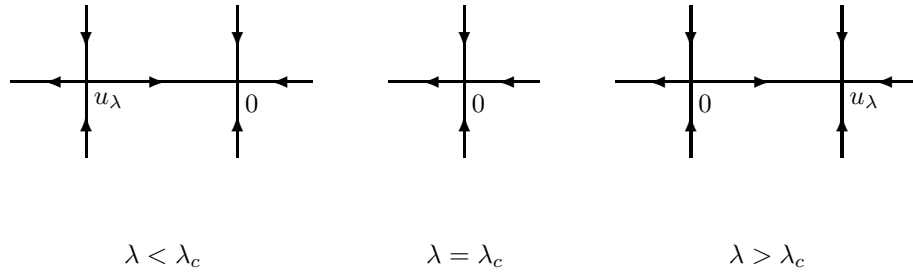


Figure3. Topological structure of dynamic bifurcation of the GSH equation. The horizontal line represents the center manifold.

4. Bifurcation of Periodic Solutions of The Swift-Hohenberg equation.

4.1. **n -dimensional Swift-Hohenberg equation.** The n -dimensional Swift-Hohenberg equation reads as

$$\begin{cases} \frac{du}{dt} = -(I + \Delta)^2 u + \lambda u - u^3, & x \in \Omega, \quad t > 0; \\ u(0) = u_0; \end{cases} \quad (23)$$

where $\Omega = (0, L)^n$, $1 \leq n \leq 3$. And the equation is supplemented with one of the following boundary conditions:

1. *The odd-periodic boundary condition:*

$$u(x_i, t) = u(x_i + L_i, t) \quad \text{and} \quad u(-x, t) = -u(x, t);$$

2. *The periodic boundary condition:*

$$u(x_i, t) = u(x_i + L_i, t) \quad \forall i.$$

4.2. Functional settings. In an appropriate functional setting the SH equation (23) can be expressed in the following form

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda), \\ u(0) = u_0, \end{cases} \quad (24)$$

where the operators L_λ and G are defined as follows:

$$\begin{aligned} L_\lambda &= -A + B_\lambda : H_1 \hookrightarrow H, \\ A &= (I + \Delta)^2 : H_1 \hookrightarrow H, \\ B_\lambda &= \lambda I : H_1 \hookrightarrow H, \\ Gu &= -u^3 : H_1 \hookrightarrow H. \end{aligned} \quad (25)$$

with Hilbert spaces H and H_1 :

$$\begin{aligned} H_1 &= \begin{cases} \{u \in \dot{H}_{per}^4(\Omega) | u(-x, t) = -u(x, t)\} & \text{for odd-periodic condition,} \\ \dot{H}_{per}^4(\Omega) & \text{for periodic condition;} \end{cases} \\ H &= \begin{cases} \{u \in \dot{L}_{per}^2(\Omega) | u(-x, t) = -u(x, t)\} & \text{for odd-periodic condition,} \\ \dot{L}_{per}^2(\Omega) & \text{for periodic condition.} \end{cases} \end{aligned} \quad (26)$$

The subscript "per" stands for "periodic" and the dot, $\dot{\cdot}$, means $\int_0^L f dx = 0$, for f in H^4 or L^2 . In any case H_1 is a dense and compact subspace of H , $H_1 \hookrightarrow H$.

Theorem 6. (An a priori estimate) *Assume λ_c be the first eigenvalue of $(I + \Delta)^2 : H_1 \hookrightarrow H$. Then the following a priori estimates hold for the SH equation:*

$$|u(t)|_2 \leq \begin{cases} e^{(\lambda - \lambda_c)t} |u(0)| & \text{if } \lambda < \lambda_c; \\ \frac{|u(0)|}{\sqrt{2|\Omega||u(0)|^2 t + 1}} & \text{if } \lambda = \lambda_c; \\ \max\left\{\sqrt{\frac{\lambda - \lambda_c}{|\Omega|}}, |u(0)|\right\} & \text{if } \lambda > \lambda_c. \end{cases} \quad (27)$$

Lemma 1. *Suppose $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the following:*

$$\psi'(t) \leq a\psi(t) - b\psi^2(t),$$

with $b > 0$. Then the following assertions are true:

1. *if $a < 0$ then $\psi(t) \leq \psi(0)e^{at}$,*
2. *if $a = 0$ then $\psi(t) \leq \frac{\psi(0)}{b\psi(0)t + 1}$,*
3. *if $a > 0$ then $\psi(t) \leq \max\{\psi(0), \sqrt{\frac{b}{a}}\}$.*

proof of theorem 6. Using the energy method, from (23) we will have:

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 = - \langle (I + \Delta)^2(u), u \rangle + \lambda_c |u|_2^2 - |u^2|_2^2.$$

It is known that the principal eigenvalue of $A = (I + \Delta)^2 : H_1 \hookrightarrow H$ satisfies

$$\lambda_c = \min_{\substack{u \in H_1 \\ u \neq 0}} \frac{\langle (I + \Delta)^2(u), u \rangle}{|u|_2^2};$$

also by Hölders inequality :

$$|u|_2^2 = \int_{\Omega} u^2 dx \leq (|\Omega|)^{\frac{1}{2}} \left(\int_{\Omega} (u^2)^2 dx \right)^{\frac{1}{2}},$$

that is

$$-|u^2|_2^2 \leq -\frac{1}{|\Omega|} |u|_2^4.$$

So we will have

$$\frac{d}{dt} |u|_2^2 \leq 2(\lambda_c - \lambda) |u|_2^2 - \frac{2}{|\Omega|} |u|_2^4.$$

Therefore 27 follows from the lemma(1). \square

By the above theorem we know that $u = 0$ is globally asymptotically stable for $\lambda \leq \lambda_c$. In fact, the existence of the global attractor for the SH equation can be proven in the same fashion as in [14].

4.3. Bifurcation of odd-periodic solutions. Assume λ_c denotes the first eigenvalues of $(I + \Delta)^2 : H_1 \hookrightarrow H$. Obviously $\lambda_c = \inf\{P(\frac{\pi}{L}K) | K \in \mathbb{Z}^n\}$, where $P(x) = (1 - |x|^2)^2$, $x \in \mathbb{R}^n$.

Theorem 7. *The following assertions are true for the SH equation (23) with the odd periodic boundary condition :*

1. *For $\lambda \leq \lambda_c$, $u = 0$ is globally asymptotically stable.*
2. *for $\lambda > \lambda_c$, the SH equation (23) bifurcates from $(u, \lambda) = (0, \lambda_c)$ to an attractor \mathcal{A}_λ which is homologic to S^{n-1} (if $n = 2$, \mathcal{A}_λ is homeomorphic S^1).*
3. *The attractor \mathcal{A}_λ contains exactly 2^n steady state solutions of (23), which are regular.*
4. *There is an $\varepsilon > 0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor \mathcal{A}_λ bifurcated from $(0, \lambda_0)$ attracts all bounded sets in H/Γ in H , where Γ is the stable manifold of $u = 0$ with codimension n .*

Proof. Without loss of generality, we assume $L \leq 2\pi$; so we have $\lambda_c = \lambda_1 = (1 - (\frac{2\pi}{L})^2)^2$. The eigenvalues and vectors of L_λ are given by the following:

$$\begin{aligned} \beta_K(\lambda) &= \lambda - (1 - (\frac{2\pi}{L})^2 |K|^2)^2, \\ \phi_K(x) &= \sqrt{\frac{2}{L^n}} \sin(\frac{2\pi}{L} K \cdot x), \end{aligned} \tag{28}$$

where $x = (x_1, x_2, \dots, x_n)$, and $K = (k_1, k_2, \dots, k_n)$.

Now let $\beta_1(\lambda) = \beta_K(\lambda)$ and $\phi_i = \phi_K$ when $K = (\delta_{i1}, \dots, \delta_{in})$. Then we have the following properties:

$$\beta_1(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_1, \\ = 0 & \text{if } \lambda = \lambda_1, \\ > 0 & \text{if } \lambda > \lambda_1, \end{cases}$$

$$\beta_K(\lambda_1) \leq 0 \quad \forall |K| \geq 2.$$

So the first part of the theorem follows from the attractor bifurcation theorem (1). In order to prove the second part of the theorem we use the Lyapunov-Schmidt reduction method; this gives us the following bifurcation equations:

$$\beta_1(\lambda) y_i - \frac{3}{2L^2} (y_i^3 + 2 \sum_{j \neq i} y_j^2 y_i) = 0. \tag{29}$$

where $1 \leq i \leq n$ and $y_i = y_K$ with $K = (\delta_{i1}, \dots, \delta_{in})$.

This equation has 2^n solutions as follows

$$|y_1| = |y_2| = \dots = |y_n| = \dots = \sqrt{\frac{2L^2\beta_1(\lambda)}{3(2n-1)}}. \quad (30)$$

It is easy to show that these solutions are regular.

For the case $n = 2$, we have $\mathcal{A}_\lambda \simeq S^1$. This follows from a similar reduction to the center manifold and theorem 3. The circle in Figure4 depicts the center manifold. The steady state points y_1 and y_3 are minimal attractors and y_2 and y_4 are saddle points. \square

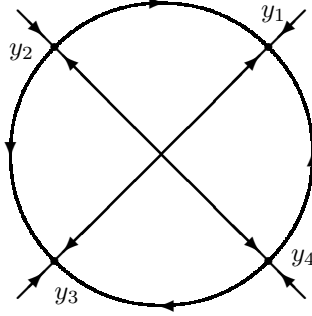


Figure4

4.4. Bifurcation Of Periodic Solutions. For the bifurcation of periodic solutions, the multiplicity of the first eigenvalue is $2n$. Hence the long time behavior of the bifurcated solutions will be essentially different from the previous case.

Theorem 8. *The following assertions are true in the case of the SH equation (23) with the periodic boundary condition :*

1. *For $\lambda \leq \lambda_c$, $u = 0$ is globally asymptotically stable.*
2. *for $\lambda > \lambda_c$, the equation bifurcates from $(u, \lambda) = (0, \lambda_c)$ to an attractor \mathcal{A}_λ which is homologic to S^{2n-1} . Moreover, when $n = 1$, \mathcal{A}_λ is homeomorphic to S^1 .*
3. *The attractor \mathcal{A}_λ contains an n -dimensional torus \mathbb{T}^n , which consist of steady state solutions of (23).*
4. *There is an $\varepsilon > 0$ such that as $\lambda_c < \lambda < \lambda_c + \varepsilon$, the attractor \mathcal{A}_λ bifurcated from $(0, \lambda_c)$ attracts all bounded sets in H/Γ in H , where Γ is the stable manifold of $u = 0$ with codimension $2n$.*

Proof. Without loss of generality, we assume $L \leq 2\pi$; so we have $\lambda_c = \lambda_1 = (1 - (\frac{2\pi}{L})^2)^2$. We get the eigenvectors and eigenvalues of $L_\lambda : H_1 \hookrightarrow H$ as follows:

$$\begin{aligned}
\beta_K(\lambda) &= \lambda - (1 - (\frac{2\pi}{L})^2 |K|^2)^2, \\
\phi_K(x) &= \sqrt{\frac{2}{L^n}} \sin(\frac{2\pi}{L} K \cdot x), \\
\psi_K(x) &= \sqrt{\frac{2}{L^n}} \cos(\frac{2\pi}{L} K \cdot x).
\end{aligned} \tag{31}$$

where $x = (x_1, x_2, \dots, x_n)$, and $K = (k_1, k_2, \dots, k_n)$.

Now assume $\beta_1(\lambda) = \beta_K(\lambda)$, $\phi_i = \phi_K$, and $\psi_i = \psi_K$, when $K = (\delta_{i1}, \dots, \delta_{in})$. Then we have the following properties:

$$\begin{aligned}
\beta_1(\lambda) &\begin{cases} < 0 & \text{if } \lambda < \lambda_1, \\ = 0 & \text{if } \lambda = \lambda_1, \\ > 0 & \text{if } \lambda > \lambda_1, \end{cases} \\
\beta_K(\lambda_1) &\leq 0 \quad \forall |K| \geq 2.
\end{aligned}$$

So by Attractor Bifurcation Theorem (1), the first assertion is true.

Assume

$$u = \sum_{i=1}^{i=n} (y_i \phi_i + z_i \psi_i) + \sum_{|K|>1}^{\infty} y_K \phi_K(x) + z_K \psi_K(x).$$

After reducing the equation SH to its center manifold, we get:

$$\begin{aligned}
\frac{dy_i}{dt} &= \beta_1(\lambda) y_i - \frac{3}{2L^2} y_i \sum_{j=1}^{j=m} (y_j^2 + 2z_j^2), \\
\frac{dz_i}{dt} &= \beta_1(\lambda) z_i - \frac{3}{2L^2} z_i \sum_{j=1}^{j=m} (z_j^2 + 2y_j^2).
\end{aligned} \tag{32}$$

for $1 \leq i \leq n$. This together with theorem (3) show that $\mathcal{A}_\lambda = S^1$ in the topological sense, when $n = 1$.

Since the subspace of the odd functions is an invariant set of $L_\lambda + G$ in H , the Lyapunov-Schmidt reduction equations of the SH equation in this subspace are the same as 29. Therefore the problem has solutions given in 30. Since the equation is invariant under the spatial translation, steady state solutions associated with 30 generate an n -dimensional torus \mathbb{T}^n as follows

$$\mathbb{T}^n = \left\{ \sqrt{\frac{2}{L^n}} \sum_{j=1}^n y_j \sin\left(\frac{2\pi}{L} x_j + \theta_j\right) + o(|y|) \mid \forall (\theta_1, \dots, \theta_n) \in \mathbb{R}^n \right\},$$

where $y = (y_1, \dots, y_n)$. This completes the proof. \square

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